Hyperbolic disordered ensembles of random matrices

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Using the simple procedure, recently introduced, of dividing Gaussian matrices by a positive random variable, a family of random matrices is generated characterized by a behavior ruled by the generalized hyperbolic distribution. The spectral density evolves from the semi-circle law to a Gaussian-like behavior while concomitantly the local fluctuations show a transition from the Wigner-Dyson to the Poisson statistics. Long range statistics such as number variance exhibit large fluctuations typical of non-ergodic ensembles.

I. INTRODUCTION

In a recent paper[1], the concept of disorder has been introduced in random matrix theory (RMT) to denote ensembles of random matrices generated by superimposing an external source of randomness to the fluctuations of the Gaussian ensembles. The method generates matrices which preserve unitary invariance though at the price of introducing correlations among matrix elements. Their statistical properties are amenable to analytical derivation. Families of ensembles recently proposed in the literature fit into this category, for instance [2, 3] in the Wigner Gaussian case and, Refs. [4–6] in the case of Wishart matrices. Disordered ensembles can be considered as an instance of the so-called superstatistics[7] aside from the fact that their density distributions are normalized differently, as explained below.

As a consequence of having operating an extra source of randomness, the ergodicity property of the standard ensembles, characterized by the equivalence between spectral averages taken along the spectrum and averages extracted from the spectra of a set of matrices is broken. This creates an ambiguity in measuring its collective spectral fluctuations although for those which concern individual eigenvalues such as extreme value statistics it remains well defined. This kind of statistics has been object of many studies[8] in the last decades. In this direction, it was shown how the behavior of the largest eigenvalue described by the Tracy-Widom distribution[8] is affected by disorder[5, 9].

Despite the great success the RMT Gaussian ensemble had and has (see [10] for recent reviews), it has features of an abstract mathematical model such as, for example, an eigenvalue density defined in a compact support. Parallel to its development, there was a search for models whose randomness would be closer to physical situations. This is the case of the two-body random ensemble (TBRE) proposed in the context of the shell model of nuclear physics[11] and of the more general k-body embedded Gaussian ensembles (EGE) proposed to many-body systems[12]. In them, the Hamiltonian model is made randomic by taking the strength of the residual interaction among particles from a Gaussian distribution. These ensembles show as main features: correlations among matrix elements, Gaussian eigenvalue density and non-ergodicity[13]. Nevertheless, their

spectral fluctuations are supposed to follow, under restricted conditions that take into account the non-ergodicity ambiguity, the same fluctuations pattern of the Gaussian matrices, namely the Wigner-Dyson statistics[14]. Disordered ensembles may provide a simple way to understand features of these physical motivated ensembles.

Families of disordered ensembles have, so far, been considered mostly in association with an external randomness governed by the gamma distribution[2, 3]. This leads to ensembles in which statistics are dominated by power law behavior that implies strong non-ergodicity. It has also been considered the case of the inverse gamma distribution[5] which contains singularities. Our purpose here is to study an ensemble whose auxiliary random variable is taken from a generalized inverse Gaussian (GIG) which acting on the Gaussian matrices generates disordered matrices distributed according to a generalized hyperbolic distribution (GH). This kind of distribution has been introduced in 1977 by Barndorff-Nielsen[15] and has found since then many applications specially in finance[16].

II. DISORDERED ENSEMBLES

We start by reviewing general aspects of the formalism of the disordered ensemble of Ref. [1]. The idea is to construct matrices by the relation

$$H(\xi) = \frac{H_V}{\sqrt{\xi/\bar{\xi}}}, \tag{1}$$

where H_V is a random matrix of dimension N with joint density probability distribution

$$P_V(H_V) = \frac{1}{Z_f} \exp\left[-\frac{\beta}{2}\mathrm{tr}V(H_V)\right] \tag{2}$$

and ξ is a positive random variable with a normalized density probability distribution $w(\xi)$ with average $\bar{\xi}$ and variance σ_ξ^2. In (2), $f = N + \beta N(N-1)/2$ is the number of independent matrix elements, β is the Dyson index $\beta = 1, 2, 4$ and $V(x)$ is a confining potential which makes normalization

$$Z_f = \int dH \exp\left[-\frac{\beta}{2}\mathrm{tr}V(H)\right] \tag{3}$$

finite with respect to the measure $dH = \prod_{i=1}^{N} dH_{ii} \prod_{j>i} \prod_{k=1}^{\beta} \sqrt{2}dH_{ij}^k$. Rewritten as $H_V = H(\xi)\sqrt{\xi/\bar{\xi}}$, Eq. (1) means that the joint distribution of the matrix elements of the disordered ensemble, is obtained from Eq. (2) by averaging over the ξ variable, namely as

$$P(H) = \int d\xi w(\xi) \left(\frac{\xi}{\bar{\xi}}\right)^{f/2} \frac{1}{Z_f} \exp\left[-\frac{\beta}{2}\mathrm{tr}V\left(\sqrt{\frac{\xi}{\bar{\xi}}}H\right)\right]. \tag{4}$$

Eq. (4) translates the relation (1) among the random quantities into a relation among their density distributions functions. Despite the resemblance of Eq. (4) with distributions constructed in the context of the superstatistics formalism[7], we remark that here the two distributions are integrated normalized while in the superstatistics case the normalization is performed after the integration.

While Eq. (4) puts in evidence the correlations among matrix elements, the relation (1) which is expressed in terms of the random quantities themselves, gives a clearer picture of the model and, at the same time, provides an efficient algorithm to generate the disordered matrices. We also remark that in [1], an alternative procedure has been described in which matrices of the ensemble are generated taking into account correlations among their elements (see next section).

Turning now to eigenvalues and eigenvectors, we observe that by diagonalizing the matrices in (1), the relation

$$D(\xi) = \frac{D_V}{\sqrt{\xi/\bar{\xi}}} \tag{5}$$

is obtained, where $D(\xi)$ and D_V are diagonal matrices whose elements are the respective eigenvalues of $H(\xi)$ and H_V. Rewriting, as done with Eq. (1), Eq. (5) as $D_V = D(\xi)\sqrt{\xi/\bar{\xi}}$, the relation

$$P(E_1, ...E_N) = \int d\xi w(\xi) \left(\frac{\xi}{\bar{\xi}}\right)^{\frac{N}{2}} P_V(x_1, x_2, ...x_N) \tag{6}$$

is derived, where $x_i = \sqrt{\xi/\bar{\xi}} E_i$ and

$$P_V(x_1, ...x_N) = K_N^{-1} \exp\left[-\frac{\beta}{2}\sum_{k=1}^{N} V(x_k)\right] J(x_1, x_2, ...x_N). \tag{7}$$

In (7), K_N is a normalization constant and $J(x_1, x_2, ...x_N)$ is the eigenvalue part of the Jacobian of the transformation from matrix elements to eigenvalues and eigenvectors. From (6), statistical measures of the generalized family can be calculated by weighting, with the $w(\xi)$ distribution, the corresponding measures of the original ensemble with the eigenvalues multiplied by the factor $\xi/\bar{\xi}$.

Considering particular choices of the distribution $w(\xi)$, the most used so far both, in the Gaussian and the Wishart cases, was the gamma distribution

$$w(\xi) = \exp(-\xi)\xi^{\bar{\xi}-1}/\Gamma(\bar{\xi}) \tag{8}$$

with variance $\sigma_\xi = \sqrt{\bar{\xi}}$. Considering the case of the Wigner ensemble in which $V(H) = H^2$, when (8) is substituted into (4), the integrals are readily performed to give the joint distribution

$$P(H; \bar{\xi}) = \frac{1}{Z_f} \frac{\Gamma(\bar{\xi} + f/2)}{\Gamma(\bar{\xi})} \left(1 + \frac{\beta}{2\bar{\xi}}\mathrm{tr}V(H)\right)^{-\bar{\xi}-f/2}. \tag{9}$$

To illustrate the power-law behavior induced by the above gamma distribution, take out of the f independent elements, the density distribution of a given one denoted by h (see next section for the definition of the h variable), after integrating over the $f-1$ others, we obtain

$$p(h; \bar{\xi}) = \left(\frac{\beta}{2\pi\bar{\xi}}\right)^{\frac{1}{2}} \frac{\Gamma(\bar{\xi} + 1/2)}{\Gamma(\bar{\xi})} \left(1 + \frac{\beta}{2\bar{\xi}}h^2\right)^{-\bar{\xi}-1/2}. \tag{10}$$

Since for large $|h|$, $p_\beta(h; \bar{\xi}) \sim 1/|h|^{2\bar{\xi}+1}$, the distribution (10) does not have moments of order superior to $2\bar{\xi}$. This fact makes the value $\bar{\xi} = 1$ critical since below it (10) does not have a second moment[2]. We remark that, apart from the lack of independence, the marginal distribution of the matrix elements has the same kind of distribution, namely one with an asymptotic power-law behavior, as the ones of the ensemble of Lévy matrices[19].

Another distribution considered in [5] was the inverse distribution obtained by making $\xi = 1/\nu$ which leads to

$$w(\nu) = \frac{\bar{\nu}}{\Gamma(1/\bar{\nu})} \exp\left(-\frac{1}{\nu}\right) \nu^{-\frac{1}{\nu}-2}. \tag{11}$$

Again, the integral in (4) can be performed to give

$$P(H; \bar{\nu}) = \frac{2}{Z_f \Gamma(1/\bar{\nu})} \left(\sqrt{\frac{\bar{\nu}}{2}\beta \mathrm{tr} V(H)}\right)^{1+\frac{1}{\bar{\nu}}-f/2} K_{f/2-1/\bar{\nu}-1}\left(\sqrt{\frac{\bar{\nu}}{2}\beta \mathrm{tr} V(H)}\right) \tag{12}$$

which shows, as a drawback, the presence of a singularity at the origin.

Our present purpose is to study the model generated by taking the disorder variable out of the generalized inverse Gaussian

$$w(a, b, \lambda; \xi) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})} \xi^{\lambda-1} \exp\left[-\frac{1}{2}(\frac{a}{\xi} + b\xi)\right], \tag{13}$$

where $K_\lambda(x)$ is the modified Bessel function of the third kind. This is a three parameter probability distribution where a and b are positive and λ is real. It contains the above distributions (8) and (11) as special cases. In fact, in the limit $a \to 0$, apart from scaling, the gamma distribution, Eq. (8), is recovered while in the limit $b \to 0$ the inverse distribution is obtained. The GIG first moment is

$$\bar{\xi} = \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} \tag{14}$$

and its variance is

$$\sigma_\xi^2 = \left(\frac{b}{a}\right)\left[\frac{K_{\lambda+2}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} - \frac{K_{\lambda+1}^2(\sqrt{ab})}{K_\lambda^2(\sqrt{ab})}\right]. \tag{15}$$

By taking the ratio

$$\frac{\bar{\xi}}{\sigma_\xi} = \frac{b}{a}\sqrt{\frac{K_{\lambda+2}(\sqrt{ab})K_\lambda(\sqrt{ab})}{K_{\lambda+1}^2(\sqrt{ab})} - 1}, \tag{16}$$

we have a better parameter to understand the effect of the disorder generated by the GIG distribution. First we remark that, by changing in (16) the sign of λ, the symmetry property $K_{-\nu}(x) = K_\nu(x)$ of the Bessel functions, makes the above ratio symmetric with respect to the value $\lambda = -1$. Secondly, for large orders, the Bessel functions can be approximated by the dominant term of their uniform approximation[17]

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi}{2\nu}} \frac{\exp(-\mu\nu)}{(1+z^2)^{1/4}}, \tag{17}$$

where

$$\mu = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}. \tag{18}$$

Using this, the Bessel functions in (16) can be approximated for large λ as

$$K_\lambda(\sqrt{ab}) \sim \sqrt{\frac{\pi}{2\lambda}} \left(\frac{\lambda}{\sqrt{ab}}\right)^\lambda \exp(-\lambda), \tag{19}$$

where, in presence of one, the square of the term $\sqrt{ab}/\lambda$ has been neglected. As a consequence, with $a = b$, the ratio (16) becomes practically constant when λ increases as indeed can be seen in Fig. 1 . On the other hand, by fixing λ and the product ab, the GIG also becomes more localized as the ratio b/a decreases. as illustrated in the same Fig. 1. From this analysis, we expect larger disorder for small values of the parameter λ or large values of the ratio b/a, with the value $\lambda = -1$ playing a special role in this generalized family.

III. HYPERBOLIC DISORDERED GAUSSIAN ENSEMBLE

The Wigner ensemble is obtained by making $V(H) = H^2$ with H a Hermitian matrix. The number of independent matrix elements is of course $f = N + \beta N(N-1)/2$ and it is convenient to consider the set of f independent elements denoted by h_n with $n = 1, 2, 3, ..., f$. They are defined such that first N ones are equal to the diagonal elements, that is, $h_i = H_{ii}$. The remain ones, the off-diagonal elements, are scaled as $h_n = \sqrt{2}H_{ij}^k$ with $n = N + 1, N + 2, ..., f$. In terms of these f independent variables, the joint distribution, Eq. (4), together with Eq. (13) becomes

$$P_f(h_1, ..., h_f) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})} \int_0^\infty d\xi \xi^{\lambda-1} \left(\frac{\beta\xi}{2\pi\bar{\xi}}\right)^{f/2} \exp\left[-\frac{1}{2}(\frac{a}{\xi} + b\xi) - \sum_{n=1}^f \frac{\beta\xi h_n^2}{2\bar{\xi}}\right]. \tag{20}$$

Each h_n variable is treated equally in (20) and that is the reason to introduce this set of variables in which the inconvenient factor of $\sqrt{2}$ multiplying the off-diagonal ones has been absorbed in the new variable. The integration over the extra variable ξ makes the distribution in (20) a correlated distribution of the set of variables, but the above distribution has the important property that removing by integration any one variable, the distribution of the remnant others has the same form, namely,we have

$$P_{f-1}(h_1, ..., h_{f-1}) = \int dh_f P_f(h_1, ..., h_f). \tag{21}$$

This property is called scale-invariant occupancy of phase space[18]. Of course, it is a manifestation of the uncorrelation among the matrix elements of the original ensemble. In this way, one can expect this property to hold in general for the disordered ensembles. On

the other side, it is not expected to hold for generalizations of distributions constructed using the strict superstatistics recipe.

As it occurred with the gamma distribution and its inverse, after substituting (13) in (4), the integrals are performed and we find the joint distribution of matrix elements and the distribution of a generic matrix element h to be given, respectively, by the generalized hyperbolic distributions

$$P(a,b,\lambda;H) = \left(\frac{\sqrt{a/b}}{2\pi\bar{\xi}}\right)^{f/2} \frac{1}{\left(1 + \frac{1}{b\bar{\xi}}\sum_{i=1}^{f} h_i^2\right)^{\lambda_f/2}} \frac{K_{\lambda_f}\left[\sqrt{ab\left(1 + \frac{1}{b\bar{\xi}}\sum_{i=1}^{f} h_i^2\right)}\right]}{K_\lambda(\sqrt{ab})}, \quad (22)$$

and

$$p(a,b,\lambda;h) = \frac{(a/b)^{1/4}}{\sqrt{2\pi\bar{\xi}}} \frac{1}{\left(1 + \frac{h^2}{b\bar{\xi}}\right)^{\lambda_1/2}} \frac{K_{\lambda_1}\left[\sqrt{ab\left(1 + \frac{h^2}{b\bar{\xi}}\right)}\right]}{K_\lambda(\sqrt{ab})}, \quad (23)$$

where with $1 \leq k \leq f$, $\lambda_k = \lambda + k/2$. The scale-invariant property has been used to write Eq. (23). Comparing these GH equations with the correspondent equations for the gamma disorder, we see that the GH ones contain in themselves the gamma ones but multiplied by the Bessel function that decays exponentially for large values of $|h|$. This leads to a more regular behavior as becomes clear from the expression

$$\overline{h^n} = \frac{\left(2\bar{\xi}b/a\right)^{n/2}}{K_\lambda} \frac{\Gamma[(n+1)/2]}{\sqrt{\pi}} K_{\lambda-n/2}\left(\sqrt{ab}\right). \quad (24)$$

for the moment of arbitrary nth order of (23). From the symmetry property of the Bessel function, i.e. $K_{-\nu}(x) = K_\nu(x)$ we conclude that (23) has all moments, resulting in milder fluctuations.

Consider the identity

$$P_f(h_1,...,h_f) = \frac{P_f(h_1,...,h_f)}{P_f(h_1,...,h_{f-1})}...\frac{P_f(h_1,...,h_k)}{P_f(h_1,...,h_{k-1})}...\frac{P_2(h_1,h_2)}{P_1(h_1)}P_1(h_1) \quad (25)$$

in which by definition each ratio gives the conditional probability $p(h_k|h_1,...,h_{k-1})$ of the last element h_k in the numerator argument once the preceding $k-1$ ones have been determined. Using the property of scale-invariance, this probability can be written in terms of the Bessel functions as

$$p(h_k|h_1,h_2,...,h_{k-1}) = \frac{(a/b)^{1/4}}{\sqrt{2\pi\bar{\xi}}} \frac{\left(1 + \frac{1}{b\bar{\xi}}\sum_1^{k-1} h_i^2\right)^{\frac{\lambda_{k-1}}{2}}}{\left(1 + \frac{1}{b\bar{\xi}}\sum_1^{k} h_i^2\right)^{\frac{\lambda_k}{2}}} \frac{K_{\lambda_k}\left[\sqrt{ab\left(1 + \frac{1}{b\bar{\xi}}\sum_1^{k} h_i^2\right)}\right]}{K_{\lambda_k}\left[\sqrt{ab\left(1 + \frac{1}{b\bar{\xi}}\sum_1^{k-1} h_i^2\right)}\right]}.$$

$$(26)$$

A more compact expression is obtained doing the following

$$1 + \frac{1}{b\bar{\xi}}\sum_{i=1}^{k-1}h_i^2 + \frac{h_k^2}{b\bar{\xi}} = \left(1 + \frac{1}{b\bar{\xi}}\sum_{i=1}^{k-1}h_i^2\right)\left[1 + \frac{h_k^2}{b\bar{\xi}\left(1 + \frac{1}{b\bar{\xi}}\sum_{i=1}^{k-1}h_i^2\right)}\right] = \frac{b_{k-1}}{b}\left(1 + \frac{h_k^2}{b_{k-1}\bar{\xi}}\right),$$

(27)

where

$$b_k = b\left(1 + \frac{1}{b\bar{\xi}}\sum_{i=1}^{k}h_i^2\right),$$

(28)

with $b_0 = b$. With these definitions the probability distribution of one element h_k once $k-1$ other ones have already been sorted is

$$p(h_k|h_1, h_2, ..., h_{k-1}) = \frac{(a/b_{k-1})^{1/4}}{\sqrt{2\pi\bar{\xi}}}\frac{1}{\left(1 + \frac{h_k^2}{b_{k-1}\bar{\xi}}\right)^{\lambda_k/2}}\frac{K_{\lambda_k}\left[\sqrt{ab\left(1 + \frac{h_k^2}{b_{k-1}\bar{\xi}}\right)}\right]}{K_{\lambda_{k-1}}(\sqrt{ab_{k-1}})},$$

(29)

With k running from 2 to f, these equations form a set of univariate distributions which can sequentially be used to generate a matrix of the ensemble in which correlations among elements are taken into account.

Integrating (6) over all eigenvalues but one and multiplying by N, the eigenvalue density is expressed in terms of the Wigner's semi-circle law density $\rho_G(E) = \sqrt{2N - E^2}/\pi$ of the Gaussian ensemble[14] as

$$\rho(E) = \frac{1}{\pi}\int_0^{\xi_{max}}d\xi w(\xi)\left(\frac{\xi}{\bar{\xi}}\right)^{1/2}\sqrt{2N - \frac{\xi}{\bar{\xi}}E^2}.$$

(30)

where $\xi_{max} = 2N\bar{\xi}/E^2$. From (30), we find that at the origin

$$\rho(0) = \frac{\sqrt{2N}}{\pi}\frac{\sqrt{\bar{\xi}}}{\sqrt{\bar{\xi}}} = \rho_G(0)\frac{\sqrt{\bar{\xi}}}{\sqrt{\bar{\xi}}}$$

(31)

and, since the ratio between the two averages is less than one, the disordered density is, at the origin, smaller than the semi-circle value. When $|E| \to \infty$, $\xi_{max} \to 0$ and therefore the interval of integration in (30) collapses and a crude approximation to the integral is

$$\rho(E) \sim \exp\left(-\frac{aE^2}{2N\bar{\xi}}\right)E^{-2\lambda-3}$$

(32)

which shows that the density has Gaussian tails. In Fig.2, it is shown that with the parameter λ fixed at the value -1, the density undergoes a transition from the semi-circle to a Gaussian-like shape which becomes more and more deformed as $a = b$ decreases.

By integrating the density, Eq. (30), from the origin to a value E, we obtain the function

$$N(E) = \frac{N}{2}\left(1 - \int_0^{\xi_{max}}d\xi w(\xi)\left[1 - \frac{2}{N}N_G\left(\sqrt{\frac{\xi}{\bar{\xi}}}E\right)\right]\right),$$

(33)

where

$$N_G(E) = \frac{N}{2}\left(\arcsin\frac{E}{\sqrt{2N}} + \frac{E}{\sqrt{2N}}\sqrt{1 - \frac{E^2}{2N}}\right). \qquad (34)$$

The functions $N(E)$ and $N_G(E)$ count the average number of eigenvalues in the interval $(0, E)$ for the the disordered and the Gaussian ensembles, respectively.

To measure spectral fluctuations, the dependence on the density is removed by the transformation $x = N(E)$ which maps the actual spectrum into a new one with unit density. Starting with short range correlations, we define the spacing function that gives the probability $E(0, s)$ that the interval $(-\frac{s}{2}, \frac{s}{2})$ is empty. To calculate this function we integrate Eq. (6) over all eigenvalues outside the interval the $(-\frac{\theta}{2}, \frac{\theta}{2})$ to obtain with $s = 2N(\frac{\theta}{2})$

$$E(0, s) = \int_0^\infty d\xi\, w(\xi) E_G\left[0, 2N_G\left(\sqrt{\frac{\xi}{\xi}}\frac{\theta}{2}\right)\right], \qquad (35)$$

where for the Gaussian spacing we use the Wigner surmise for the GOE case ($\beta = 1$)

$$E_G(0, s) = \operatorname{erfc}(\sqrt{\frac{\pi}{4}}s). \qquad (36)$$

From this function the nearest neighbor distribution (NND) is derived as $p(s) = \frac{d^2 E(0,s)}{ds^2}$ which gives

$$p(s) = \begin{cases} -\frac{\rho'(\theta/2)E'(0,s)}{2\rho^3(\theta/2)} + \frac{1}{2\rho^2(\theta/2)}\int_0^{\xi_m} d\xi\, w(\xi)(\frac{\xi}{\xi})\rho'_G\left(\sqrt{\frac{\xi}{\xi}}\frac{\theta}{2}\right) E'_G\left[0, 2N_G(\sqrt{\frac{\xi}{\xi}}\frac{\theta}{2})\right] \\ +\frac{1}{\rho^2(\theta/2)}\int_0^{\xi_m} d\xi\, w(\xi)(\frac{\xi}{\xi})\rho_G^2\left(\sqrt{\frac{\xi}{\xi}}\frac{\theta}{2}\right) p_G\left[0, 2N_G(\sqrt{\frac{\xi}{\xi}}\frac{\theta}{2})\right] \end{cases} \qquad (37)$$

The above equations are exact and the spacings presented below in Fig. 3 were calculated with them. To understand these results we derive approximated spacings considering that for large N, the two counting functions, $N(E)$ and $N_G(E)$ can be replaced, at the center of the spectrum, by $N(E) = \rho(0)E$ and $N_G(E) = \rho_G(0)E$. With these approximations, Eq. (35) becomes

$$E(0, s) = \frac{(b/a)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{ab})}\int_0^\infty d\xi \exp\left[-\frac{1}{2}(\frac{a}{\xi} + b\xi)\right]\xi^{\lambda-1}\operatorname{erfc}\left[\frac{\sqrt{\xi\pi}\rho_W(0)}{2\sqrt{\bar{\xi}}\rho(0)}s\right] \qquad (38)$$

and its derivative $F(0, s) = -dE(0, s)/ds$

$$F(0, s) = \frac{(b/a)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{ab})}\frac{\rho_W(0)}{\rho(0)\sqrt{\bar{\xi}}}\int_0^\infty d\xi\, e^{-\frac{1}{2}\left[\frac{a}{\xi} + \left(b + \frac{\pi\rho_W^2(0)s^2}{2\bar{\xi}\rho^2(0)}\right)\xi\right]}\xi^{\lambda-\frac{1}{2}}. \qquad (39)$$

We remark that using Eq. (31) it can be verified that the condition $F(0, 0) = 1$ is satisfied. We can go further by expressing Eq. (39) in terms of Bessel functions as

$$F(0, s) = \frac{(b/a)^{\frac{\lambda}{2}}}{K_\lambda(\sqrt{ab})}\frac{\rho_W(0)}{\rho(0)\sqrt{\bar{\xi}}}\frac{K_{\lambda_1}\left[\sqrt{ab\alpha(s)}\right]}{[\alpha(s)]^{\lambda_{1/2}}}, \qquad (40)$$

where

$$\alpha(s) = 1 + \frac{\pi \rho^2(0)}{2b\rho_G^2(0)\bar{\xi}}s^2.$$ (41)

Finally, in the same approximation the NND has the expression

$$p(s) = \frac{K_\lambda^2(\sqrt{ab})}{K_{\lambda_1}^3(\sqrt{ab})} \frac{K_{\lambda_3}\left[\sqrt{ab\alpha(s)}\right]}{[\alpha(s)]^{\lambda_3/2}} \frac{\pi}{2}s.$$ (42)

Replacing now the Bessel function by its asymptotic behavior, assuming that as a function of s, its argument is large, we find the NND decays as

$$p(s) \sim \exp\left(-\sqrt{ab\left[1 + \frac{\pi\rho^2(0)}{2b\rho_G^2(0)\bar{\xi}}s^2\right]}\right).$$ (43)

This decaying can present two limiting situations, first, if a and b increases, the second term inside the square root becomes smaller than one, in such a way that a Gaussian decay is obtained by expanding the square root. In the second situation, b decreases and makes the second term much greater than one such that it becomes the dominant term leading to an exponential decay.

Therefore, as a function of its parameters, this model constitutes a family which locally describes intermediate cases between the Wigner-Dyson and the Poisson statistics. This is illustrated in Fig. 3 where the cumulative NND, $F(s) = 1 - F(0, s)$, is plotted by the indicated values of the parameters. We see that as the density goes from the semi-circle to the Gaussian-like shape, concomitantly the spacing moves from the Wigner surmise to the Poisson distribution $1 - \exp(-s)$. Further, at an intermediate point of the transition, which coincides with the density approaching a Gaussian shape, the cumulative NND approaches the so-called semi-Poisson distribution given by $1 - (1 + 2s)\exp(-2s)$[20].

A statistics that measures spectral long range correlations and, also estimates non-ergodicity[1, 21], is the variance $\Sigma^2(L)$ of the number of eigenvalues in the interval $[-\theta/2, \theta/2]$ with $L = 2N(\theta/2)$. In [1] it is shown that this variance is given by

$$\Sigma^2(L) = \int d\xi w(\xi) \left[\Sigma_G^2\left(2N_G(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2})\right) - 2N_G\left(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2}\right) + 4N_G^2\left(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2}\right)\right] + L - L^2.$$ (44)

where Σ_G^2 is Gaussian number variance. In the limit of large spectra, we again use the linear approximations of the two counting functions to obtain

$$\Sigma^2(L) == \Sigma_G^2\left[\frac{\rho_G(0)}{\rho(0)}L\right] + \left[\frac{\rho_G(0)}{\rho(0)} - 1\right]L^2,$$ (45)

where using the fact that $\Sigma_G^2(x)$ is a logarithmic smooth function of its argument, the integral was asymptotically performed. Eq. (45) shows that the effect of disorder in the number variance statistics is twofold: (i) it enhances it by rescaling the argument of the Gaussian expression and (ii) it introduces a superpoissonian quadratic term that affects large intervals. These effects are illustrated in Fig. 4, in which Eq. (45) is calculated

with parameters values which give a density and a spacing distribution closest to the Wigner-Dyson's ones in Fig.2 and 3.

To study the behavior of the largest eigenvalues in the limit of large matrix size N, one introduces the scaled variable

$$s(E) = \sqrt{2}N^{2/3}\left[\frac{E}{\sqrt{2N}} - 1\right]. \tag{46}$$

in terms of which the probability $E_{G,\beta}(0, s)$ that the infinite interval (s, ∞) is empty is known for the three symmetry classes[8]. For the unitary class, $\beta = 2$

$$E_{G,2}(s) = \exp\left[-\int_s^\infty (x - s)q^2(x)dx\right] \tag{47}$$

where $q(s)$ satisfies the Painlevé II equation

$$q'' = sq + 2q^3 \tag{48}$$

with boundary condition

$$q(s) \sim \text{Ai}(s) \text{ when } s \to \infty, \tag{49}$$

where $\text{Ai}(s)$ is the Airy function. For the orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) classes the probabilities are given, respectively, by

$$[E_{G,1}(s)]^2 = E_2(0, s)\exp\left[\mu(s)\right] \tag{50}$$

and

$$[E_{G,4}(s)]^2 = E_2(0, s)\cosh^2\frac{\mu(s)}{2} \tag{51}$$

where

$$\mu(s) = \int_s^\infty q(x)dx. \tag{52}$$

The above equations give a complete description of the fluctuations of the eigenvalues at the edge of the spectra of the Gaussian ensembles. When perturbed, it has been shown[9] that for the three symmetry classes the above probabilities modify to

$$E_\beta(\lambda_{max} < t) = \int d\xi w(\xi)E_{G,\beta}[S(\xi, t)] \tag{53}$$

with the argument of $S(\xi, t)$ obtained by plugging in the above ξ-variance namely

$$S(\xi, t) = \sqrt{2}N^{2/3}\left[\frac{t}{\sqrt{2N}}\sqrt{\frac{\xi}{\bar{\xi}}} - 1\right]. \tag{54}$$

Equations (53) and (54) give a complete analytical description of the behavior of the largest eigenvalue once the function $w(\xi)$ is chosen. In [9], asymptotic results, when the limit $N \to \infty$ is taken, have been derived without specifying $w(\xi)$. To do this, it was assumed that the localization of $w(\xi)$, given by the ratio $\sigma_w/\bar{\xi}$ was dependent on the

matrix size N. Considering the distribution $w(\xi)$ independent of N keeping the ratio $\frac{t}{\sqrt{2N}}$ fixed when the matrix size N increases, for the three invariant ensembles, the function $E_{G,\beta}$ becomes a step function centered at $\xi = 2N\bar{\xi}/t^2$. Therefore, in this regime, the probability distribution for the largest eigenvalue converges to

$$E_\beta\left(\lambda_{max} < t\right) = \int_{2N\bar{\xi}/t^2}^{\infty} d\xi w\left(\xi\right) \tag{55}$$

with density

$$\frac{dE_\beta\left(t\right)}{dt} = \frac{4N\bar{\xi}w(2N\bar{\xi}/t^2)}{t^3}. \tag{56}$$

that shows a GIG distribution for the square of the largest eigenvalue.

IV. HYPERBOLIC DISORDERED WISHART MATRICES

Taking now $V = X^\dagger X$ with X being a rectangular matrix X of size $(M\text{x}N)$, the Wishart ensemble is defined by the joint density distribution

$$P_W(X) = \left(\frac{\beta}{2\pi}\right)^{f/2} \exp\left(-\frac{\beta}{2}\text{tr}(X^\dagger X)\right), \tag{57}$$

where $f = \beta MN$. From (57), the elements of X are Gaussian distributed. The eigenvalues of the random matrix $V = X^\dagger X$ have joint density distribution

$$P_W(x_1, x_2, ..., x_N) = K_N \exp(-\frac{\beta}{2}\sum_{k=1}^{N} x_k)\prod_{i=1}^{N} x_i^{\frac{\beta}{2}(1+M-N)-1}\prod_{j>i} \mid x_j - x_i \mid^\beta. \tag{58}$$

This ensemble was introduced by the statistician J. Wishart[22] and plays an important rôle in statistical analysis[23] where $V = X^\dagger X$ is a covariant matrix. More recently, it appeared associated in the chiral random ensemble[24] when the square of its eigenvalue is taken.

It can be shown that, for large matrices, the density of these eigenvalues approaches the Marchenko-Pastur density[25]

$$\rho_{MP}(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}, \tag{59}$$

in which, with $c = \sqrt{\frac{M}{N}}$, $x_\pm = N(c \pm 1)^2$. The counting function obtained integrating this density is

$$N_W(x) = \frac{1}{4\pi}\left[\begin{array}{c} -4\sqrt{x_-x_+}\arctan\sqrt{\frac{x_+(x-x_-)}{x_-(x_+-x)}} + (x_+ + x_-)\arccos\left(\frac{x_++x_--2x}{x_+-x_-}\right) \\ +2\sqrt{(x_+ - x)(x - x_-)} \end{array}\right. . \tag{60}$$

As in the Wigner case, disorder is introduced in the ensemble by defining new matrices through the relation

$$X(\xi) = \frac{X_W}{\sqrt{\xi/\bar{\xi}}}, \tag{61}$$

which replaced in (57) leads to an ensemble with joint density distribution of matrix elements

$$P(X) = \int d\xi w(\xi) \left(\frac{\beta\xi}{2\pi\bar{\xi}}\right)^{f/2} \exp\left(-\frac{\beta\xi}{2\bar{\xi}}\mathrm{tr}(X^\dagger X)\right). \tag{62}$$

After substituting (13) in (62), as it occurred in the Wigner case, integrals can be performed and we get

$$P(a, b, \lambda; H) = \left(\frac{\sqrt{a/b}}{2\pi\bar{\xi}}\right)^{f/2} \frac{1}{\left(1 + \frac{\mathrm{tr}(X^\dagger X)}{b\xi}\right)^{\lambda_f/2}} \frac{K_{\lambda_f}\left[\sqrt{ab\left(1 + \frac{\mathrm{tr}(X^\dagger X)}{b\xi}\right)}\right]}{K_\lambda(\sqrt{ab})}. \tag{63}$$

The joint distribution of eigenvalues is given by Eq. (6) with P_V replaced by P_W. Eigenvalue measures are therefore obtained by averaging those of the Wishart matrices and, for instance, the eigenvalue density is given by

$$\rho(x) = \frac{1}{2\pi x} \int_{\xi_-}^{\xi_+} d\xi w(\xi) \left(\frac{\xi}{\bar{\xi}}\right)^{1/2} \sqrt{\left(x_+ - x\sqrt{\frac{\xi}{\bar{\xi}}}\right)\left(x\sqrt{\frac{\xi}{\bar{\xi}}} - x_-\right)}, \tag{64}$$

where

$$\xi_\pm = \bar{\xi}\left(\frac{x_\pm}{x}\right)^2. \tag{65}$$

Of course, the above density extends beyond the Marchenko-Pastur limits $x_\pm$ as can be seen in Fig. 5. By taking the limit $x \to \infty$, the integration interval collapses and a crude approximation to the integral gives the exponential decay

$$\rho(x) \sim \exp\left[-\frac{ax^2}{2\bar{\xi}(x_+^2 + x_-^2)}\right]/x. \tag{66}$$

For the counting function we find after integrating the density

$$N(x) = \int_{\xi_-}^{\xi_+} d\xi w(\xi) N_W\left(\sqrt{\frac{\xi}{\bar{\xi}}}x\right) + N\left[1 - \int_0^{\xi_+} d\xi w(\xi)\right]. \tag{67}$$

The Wishart ensemble belongs to a class of random ensembles whose spectral fluctuations are derived from a general formalism based on orthogonal polynomials, the Laguerre ones, and share the same universal statistiscal properties[26]. This universality in the case of the Wigner-Dyson statistics is manifested after mapping the eigenvalues into variables with density one. We consider two spacing functions which give the probability of an interval to be empty: one at the bulk and the other at the inferior extreme of the spectrum.

At the bulk, we take the interval $(\bar{x} - \theta/2, \bar{x} + \theta/2)$ where $\bar{x} = M$ is the average position of the Marchenko-Pastur density. The spacing or gap function after introducing disorder is related to the unperturbed one by

$$E(0,s) = \int_0^\infty d\xi w(\xi) \left(E_G \left[0, N_W \left(\sqrt{\frac{\xi}{\bar{\xi}}} (M + \frac{\theta}{2}) \right) \right] - E_G \left[0, N_W \left(\sqrt{\frac{\xi}{\bar{\xi}}} (M - \frac{\theta}{2}) \right) \right] \right), \tag{68}$$

where the spacing s is calculated with Eq. (67) as

$$s = N \left(x = M + \frac{\theta}{2} \right) - N \left(x = M - \frac{\theta}{2} \right). \tag{69}$$

Comparing (68) and (35), we conclude that these spacings have the same behavior.

To study the behavior of the smallest eigenvalues in the limit of large matrix size N, one introduces the scaled variable $s = x/4N$ in terms of which the probability $E_{W,\beta}(0,s)$ that the interval $(0,s)$ is empty has been derived in Refs. [27, 28] for $\beta = 2$ and [29] for $\beta = 1$. When perturbed, the above probabilities modify to

$$E(0,s) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})} \int_0^\infty d\xi \xi^{\lambda-1} \exp\left[-\frac{1}{2}(\frac{a}{\xi} + b\xi) \right] E_{W,\beta} \left(s\sqrt{\frac{\xi}{\bar{\xi}}} \right), \tag{70}$$

which shows how the smallest eigenvalue distribution is deformed in the presence of a hyperbolic disorder. This probability becomes simple when $M = N$ with the Marchenko-Pastur density diverging at the origin. In this case the probability $E_{W,\beta}(0,s)$ assume simple exponential form particularly for $\beta = 1$.

V.   CONCLUSION

We have investigated the effect of superimposing an extra source of randomness governed by a generalized inverse Gaussian to the Gaussian fluctuations of the Wigner and the Wishart ensembles. The result is an ensemble of random matrices ruled by the generalized hyperbolic distribution which contains as particular cases disordered ensembles previously studied. The spectral density and short range statistics like spacing distribution show a transition from Wigner-Dyson to Poisson statistics, approaching universal critical statistics, namely semi-Poisson statistics, at an intermediate point of the transition. However, differently from the short range, long range statistics of the hyperbolic ensemble show a superpoissonian behavior with large fluctuations. The combination of semi-Poisson at short range and super-Poisson at long range have been observed in the non-ergodic embedded Gaussian models of many-body systems[13]. Therefore the present hyperbolic model is more suited to give a simple way to model features associated with the non-ergodicity of physical motivated ensembles.

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Figure Captions

Fig. 1 The GIG distribution, Eq. (13), is plotted for the indicated values of the

parameters: in a) the parameters a and b are kept constants while in b) $\lambda = 5$ and the ratio a/b is varied keeping $ab = 4$.

Fig. 2 The transition of the density from the semi-circle to a Gaussian-like for the indicated values of the parameters.

Fig. 3 The transition of the spacing $F(s)$ from Wigner-Dyson to Poisson statistics is shown for the same values of the parameters of Fig. 2.

Fig. 4 The nearest-neighbor distribution (NND) calculated with the approximated equation (42) for the same values of the parameters of Figs. 2 and 3.

Fig. 5 The number variance with small level of disorder is shown and compared with the GOE, the Poisson and the semi-Poisson predictions.

Fig. 6 The density of the disordered Wishart matrices of size M=80 and N=40 is compared with the Marchenko-Pastur density for the indicated values of the parameters.